PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. Consider the following game G.

		Player 2		
		L	C	R
	U	6, 3	3, 0	0, 1
Player 1	M	5, 11	1, 7	10, 10
	D	2, 5	2, 7	0, 0

(a) Solve the game by iterated elimination of strictly dominated strategies. If you get a unique solution, indicate this. If your solution is not unique, write up the reduced game where you have eliminated the strictly dominated strategies.

Solution: For player 2, R is strictly dominated by L. After eliminating R, M and D are strictly dominated by U for player 1. After eliminating M and D, C is strictly dominated by L. The unique solution is (U, L).

(b) Find all the pure and mixed-strategy Nash Equilibria.

Solution: By the results on the connection between IESDS and NE, a unique IESDS solution must also be the unique NE. Therefore, (U, L) is the unique NE.

(c) What are the Subgame-perfect Nash Equilibria of the game $G(2)$, i.e. the game G repeated twice?

Solution: Since there is only stage-game NE, the unique SPNE of $G(2)$ is to play (U, L) in all subgames.

(d) Now consider the game $G(\infty, \delta)$, i.e. the game G repeated infinitely many times, with discount factor $\delta \in (0, 1)$. Define average payoffs as

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^t \pi_{it},$$

where π_{it} is the stage-t payoff of player $i = 1, 2$. What does the *folk theorem* tell us about the set of possible equilibrium payoffs in this game as δ grows large? Show that there is a subgame-perfect equilibrium that achieves 10 as average payoffs for both players.

Solution: The folk theorem (in the version seen in class), tells us that if x is a feasible payoff (i.e. x is a convex combinations of payoffs in G) and $x_1 \geq 6$ and $x_2 \geq 3$ (strict inequalities are ok as well) then if δ is sufficiently close to 1, there exists a SPNE that achieves x as the average payoff.

Existence of the SPNE. Strategies: Play (M, R) as long as there has been no deviation. Following a deviation, play (U, L) forever. Average payoffs on the equilibrium path are $(1 - \delta) \sum_{t=1}^{\infty} \delta^t (10) = 10$, whereas after a deviation they are 6 for player 1 and 3 for player 2. Clearly, player 1 has no incentive to deviate, since his highest deviation payoff is 0 and $10 \geq (1 - \delta)(0) + \delta 6$. Player 2's best deviation payoff is 11, so he will not deviate if $10 \geq (1 - \delta)(11) + \delta(3)$, which translates into $\delta \geq \frac{1}{8}$. Thus, for δ higher than 1/8, the equilibrium can be sustained. Off the equilibrium path, no player has an incentive to deviate, since a stage-game NE will be played in every round.

2. Country A produces oil of a value of \$10 a year. However, in order to get the oil to the market, country A will have to build a pipeline through either country B or country C. Neither country B nor C are oil-producers, but they have to give permission to A in order for the pipeline to be built, and they can demand a payment from A for this. Constructing a pipeline is otherwise costless.

(a) Suppose first that all the oil can be transported by a single pipeline. Let us think of this as a cohesive coalitional game with transferable payoffs. Why can we think of this game as cohesive? Why can we think of it as having transferable payoffs?

Solution: The game is cohesive because everything that can be achieved by a coalition of one or two countries can also be achieved by all three together. The game has transferable payoffs because the countries can make monetary transfers.

(b) Write up the value of the different coalitions and find the core of the coalitional game described in (a). Give an intuition for the outcome.

Solution: The values of the different coalitions are

$$\begin{aligned} v(i) &= 0 \text{ for all } i = A, B, C \\ v(Aj) &= 10 \text{ for all } j = B, C \\ v(BC) &= 0 \\ v(ABC) &= 10. \end{aligned}$$

In the core, $\sum_{i \in I} v_i \geq v(I)$ for all coalitions I and feasibility $v_A + v_B + v_C \leq 10$. We thus have $v_i \geq 0$ for all i, $v_A + v_B \geq 10$ and $v_A + v_C \geq 10$, and from feasibility $v_A + v_B + v_C = 10$. Therefore, the core is $\{(10, 0, 0)\}$. Intuition: B and C can offer exactly the same to A, but there is no gain to making an agreement with both of them. Therefore, A has all the 'bargaining power'.

(c) Now, suppose that (i) B can transport all the oil, but C can at most transport half of it, and (ii) A can transport all the oil via an alternative route, but at a cost of \$5. Write up the value of the different coalitions and find the core of this coalitional game. How does the outcome differ from the outcome you found in (b)? Give an intuition.

Solution: The values of the different coalitions are

$$\begin{aligned} v(i) &= 0 \text{ for all } i = B, C \\ v(A) &= 5 \\ v(AB) &= 10 \\ v(AC) &= 5 \\ v(BC) &= 0 \\ v(ABC) &= 10. \end{aligned}$$

In the core, $\sum_{i \in I} v_i \geq v(I)$ for all coalitions I and feasibility $v_A + v_B + v_C \leq 10$. We thus have $v_B, v_C \geq 0$, $v_A \geq 5$, $v_A + v_B \geq 10$, $v_A + v_C \geq 5$, and from feasibility $v_A + v_B + v_C = 10$. This yields $v_A \geq 5$, $v_B \leq 5$ and $v_C \leq 0$. Therefore, the core is $\{(v_A, v_B, v_C) : v_A \in [5, 10], v_B = 10 - v_A, v_C = 0\}$. Intuition: Now, B can always offer something better than C, who therefore gets nothing. On the other hand, A has an 'outside option', and is therefore assured at least 5. The remaining surplus is bargained between A and B.

3. In this question, we consider two games in which player 1 takes an action and also chooses whether that action is observable to player 2. These two games are described by Figure 1 and Figure 2, respectively.

   The only difference between the games is the order in which player 1 moves: in the first game (Figure 1), player 1 first chooses whether the action is observable, and then takes the action. In the second game (Figure 2), player 1 first takes the action, and then chooses whether the action will be observed. Notice from the payoffs in the game trees that player

1 must invest 1 in order to make his action observable (i.e. his payoff is lower by 1 whenever he makes the action observable).

(a) Consider the game in Figure 1. Here, *player 1 has to decide whether to make his action observable **before** he takes the action.* How many proper subgames are there in this game (not counting the game itself)? What are the strategy sets of each player?

Solution: There are 4 subgames not counting the game itself (one after N, one after I, one after I and L', one after I and R'). $S_1 = \{I, N\} \times \{L, R\} \times \{L', R'\}$ and $S_1 = \{l, r\} \times \{l', r'\} \times \{l'', r''\}$.

(b) Find all the pure-strategy **Subgame-perfect Nash Equilibria** of the game in Figure 1.

Solution: Start in the subgame after N. Here, the unique NE is (R, r). In the subgame after I and L' the NE is to play l', whereas in the subgame after I and R' the NE is to play r''. Finally, in the subgame after I, the only NE which is also an NE in the two following subgames, is $(L', l'r'')$. Thus, in his first information set, player 1 is choosing between a payoff of 2 if he plays N, and a payoff of 4 if he plays I. It follows that the unique SPNE is $(IRL', rl'r'')$.

(c) Consider the equilibrium (equilibria) you found in question (b). If player 1 chooses to reveal his action (plays I), give an intuition for why this is the case. If player chooses not to reveal his action (plays N), give an intuition for why this is the case. Make the connection with the idea of commitment.

Solution: Player 1 chooses to make his action observable, since otherwise he cannot commit to not playing R which is a dominant strategy for him. By making his action observable, he effectively allows player 2 to 'punish' him for playing R by playing r. This then makes L the optimal action for player 1 after I. Thus, making his action observable in some sense gives him 'commitment power'.

(d) Now consider the game in Figure 2. Here, *player 1 has to decide whether to make his action observable **after** he takes the action.* Argue that Subgame-perfect Nash Equilibrium is not a good solution concept to solve this game. Show that there is a Perfect Bayesian Equilibrium in which the players get payoffs $(4, 5)$. Remember to specify full strategies and the beliefs that support the equilibrium.

Solution: SPNE is not a good solution concept in this case, since player 2's non-singleton information set cannot be analyzed separately. That is, this information sets is not part of any proper subgame. This means it is not possible to first solve for Player 2's action in this subgame, and then move backwards up the game tree.

PBE: The equilibrium payoffs will be achieved if player 1 plays L and then I, and player 2 then plays l. It is easy to check that player 2's best responses in his singleton information sets are l and r''. In the non-singleton information set, player 2's best response is r' if $p \leq 3/8$, and l' if $p \geq 3/8$.

Now turn to player 1. He will choose I after L if player 2 plays r' in his non-singleton information set. Thus, we require $p \leq 3/8$. Since we are looking for an equilibrium where player 1 chooses I and then L, this belief is off the equilibrium path, and can be freely specified (R4 does not apply here). If player 2 plays r', this will make player 1 choose N' after R. Finally, in his first move player 1 is therefore choosing between payoff 2 (from R) and payoff 4 (from L), and will choose L. Thus, the PBE is $(LIN', lr'r''; p \leq 3/8)$.

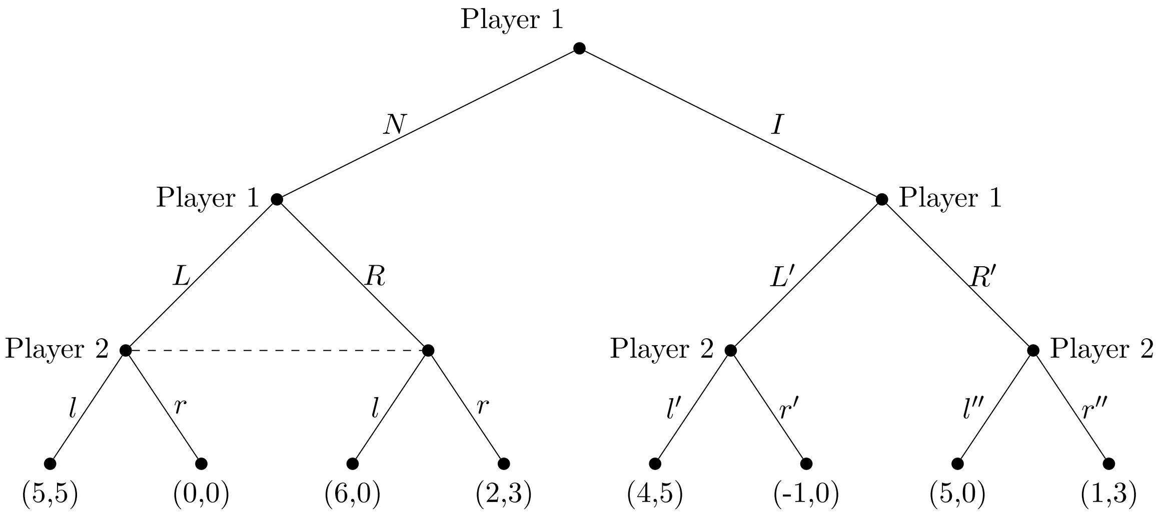
Player 1
N
I
Player 1
Player 1
L
R
L′
R′
Player 2
Player 2
Player 2
l
r
l
r
l′
r′
l″
r″
(5,5)
(0,0)
(6,0)
(2,3)
(4,5)
(-1,0)
(5,0)
(1,3)

FIGURE 1

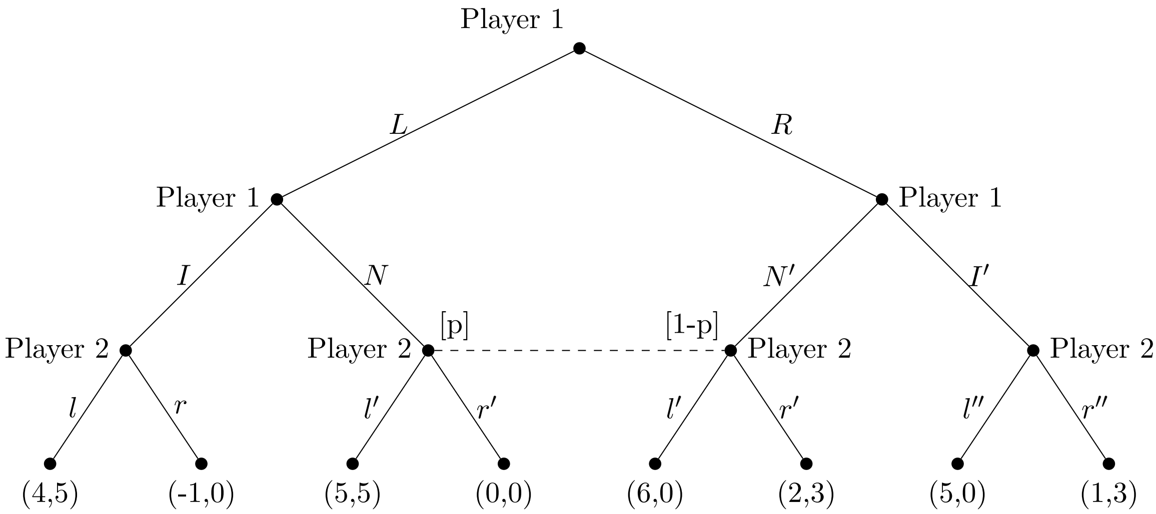
Player 1
L
R
Player 1
Player 1
I
N
N′
I′
[p]
[1-p]
Player 2
Player 2
Player 2
Player 2
l
r
l′
r′
l′
r′
l″
r″
(4,5)
(-1,0)
(5,5)
(0,0)
(6,0)
(2,3)
(5,0)
(1,3)

Figure 2